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Geometry of the Jelonek set[☆]

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Abstract

Let \mathbb{K} be an algebraically closed field of an arbitrary characteristic. In this paper, we show that the Jelonek set of a polynomial generically finite map $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ (i.e. the set of points at which the map f is not finite) is a \mathbb{K} -uniruled variety of pure dimension $n - 1$ or the empty set. We also give an example that it is not necessarily separably uniruled although the map is separable.

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1. Introduction

In the early 1990s Zbigniew Jelonek started to study some nice geometrical properties of polynomial mappings, which can help in better understanding of the problem of the Jacobian Conjecture. More precisely he proposed to describe the set of points at which a given polynomial mapping is not proper. (We say that the map $f : X \rightarrow Y$ is proper at the point $y \in Y$ if there exists an open neighborhood of y such that the restriction map $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is proper.) In paper [2], he proved that for a dominant polynomial map $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$ this set is a hypersurface. Later he generalized this result for semi-affine

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varieties over \mathbb{C} . Let us recall that an irreducible algebraic variety X is said to be *semi-affine* if there exist an affine variety W and a proper, generically finite map $\varphi : X \rightarrow W$ (we say that a polynomial map $f : X \rightarrow Y$ is generically finite if there exists an open Zariski dense subset $U \subset X$ such that for all points $x \in U$ the fibers $f^{-1}(f(x))$ are finite). Jelonek showed that in the case where X is dominated by \mathbb{C}^n the set of points at which a polynomial dominant mapping $f : X \rightarrow Y$ is not proper is a \mathbb{C} -uniruled hypersurface (i.e. it is dominated by a cylinder), see [4, Theorem 5.7]. Nowadays, properties of the set investigated by Jelonek begin to play an important role in many geometrical problems, see for example [3,6]. Though there are many properties which can be translated directly to the case of arbitrary characteristic there arise some natural difficulties which demand to use other technics to prove them.

In the sequel we work over an algebraically closed field \mathbb{K} of any characteristic. Following some ideas of Jelonek we give an analog description of the set of points at which a polynomial mapping is not finite in the case of an arbitrary algebraically closed field. More precisely, we have the following pure algebraic definition.

Definition 1.1. Let $X \subset \mathbb{K}^n, Y \subset \mathbb{K}^m$ be affine irreducible varieties. We say that the dominant polynomial map $f : X \rightarrow Y$ is finite at a point $y \in Y$ if there exists a Zariski open neighborhood U of y such that the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is finite.

Recall that by finite map of affine varieties, we mean a map $f : X \rightarrow Y$ such that an induced extension of rings of coordinates $f^*(\mathbb{K}[Y]) \subset \mathbb{K}[X]$ is integral and by dominant map we mean such a map that its image is dense in the target space. The set of points at which a polynomial map f is not finite is denoted by J_f , and called the Jelonek set. In his latest paper of the series [5], Jelonek proved that the set J_f for a generically finite map $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is of pure dimension $n - 1$ or the empty set.

In this note we prove that for such a map the Jelonek set is also \mathbb{K} -uniruled (Theorem 4.1).

We organize this paper as follows. At the beginning we establish some auxiliary results which we will use to prove the main theorem. In the third part, we give a characterization of a \mathbb{K} -uniruled variety. Actually Theorem 3.1 is an affine counterpart of the projective version of Proposition IV.1.3 from the book [8]. This part can be skipped directly to the next part where we give a proof of the Theorem 4.1. The last section is devoted to a study of separable mappings, it means such dominant maps $f : X \rightarrow Y$ of affine irreducible varieties that the associated field extension $f^*(\mathbb{K}(Y)) \subset \mathbb{K}(X)$ is separable. At the end we show that in positive characteristic strange things can happen, namely, we give an example of quasi-finite (i.e. with all fibers finite) separable map $f : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ such that the Jelonek set J_f is not separably \mathbb{K} -uniruled. All maps considered in this paper are supposed to be polynomial. We denote by \mathbb{P}^n the projective space of dimension n over the ground field \mathbb{K} and by $\mathbb{P}(V)$ projectivization of the variety V .

2. Auxiliary results

At the beginning let us recall two versions of the Zariski's Main Theorem which we will use.

Theorem 2.1 (Zariski's Main Theorem). *Let X, Y be irreducible varieties, and assume that Y is normal. Let $\phi : X \rightarrow Y$ be a proper birational morphism, then for every $y \in Y$ the fiber $\phi^{-1}(y)$ is connected.*

Theorem 2.2 (Grothendieck's version). *Let X, Y be affine varieties and let $f : X \rightarrow Y$ be a quasi-finite map. Then there exist an affine variety V which contains X as an open dense subset and a finite map $F : V \rightarrow Y$ such that $F|_X = f$.*

The following observation due to Jelonek [4, Lemma 4.5] will be very useful in the proof of Theorem 3.1.

Proposition 2.1. *Let X be an irreducible complete variety of dimension at least 2, and $Y \subset X$ an open semi-affine subvariety. Then the set $X \setminus Y$ is connected.*

Even though the given proof by Jelonek is for characteristic 0 it works also in positive characteristic.

We will need also the following fact.

Proposition 2.2. *Let X be a semi-affine irreducible normal surface containing \mathbb{K}^2 as a proper subset. Then $X \setminus \mathbb{K}^2$ is a finite union of parametric curves.*

Proof. By assumption \mathbb{K}^2 is an open and dense subset of X hence $X \setminus \mathbb{K}^2$ is of pure codimension 1, and in consequence it is a finite sum of irreducible curves, say Γ_i . We will show that all its connected components are images of rational maps defined on \mathbb{K} . Let us consider a projective closure \bar{X} of the surface X and a rational map $\iota : \mathbb{P}^2 \dashrightarrow \bar{X}$, which is an extension of the inclusion $\mathbb{K}^2 \subset X$. If ι is a regular map then since $\mathbb{P}^2 \setminus \mathbb{K}^2$ is irreducible we have $\iota(\mathbb{P}^2 \setminus \mathbb{K}^2) = \bar{X} \setminus X$ and finally $X = \mathbb{K}^2$.

In the other case, ι has only finitely many points of indeterminacy. By resolving it in these points (like in [12, IV,3]) there exist a variety Y and a regular map $\phi : Y \rightarrow \bar{X}$ such that the following diagram is commutative:

$$\begin{array}{ccc} Y & & \\ \pi \downarrow & \searrow \phi & \\ \mathbb{P}^2 & \xrightarrow{\iota} & \bar{X} \end{array}$$

where π is a composition of blowing-ups. Notice that all centers of these are at infinity. Let $Z := E_0 \cup E_1 \cup \dots \cup E_m \subset Y$ be a tree coming from these blowing-ups. Let E_0 be an infinity line, and E_i exceptional divisors for $i = 1, \dots, m$. Since the complement of an affine open dense set in the projective variety is a connected hypersurface ([1, p. 67, Proposition]) we get that $\bar{X} \setminus X$ is connected. Thanks to the Zariski's Main Theorem the fibers of the morphism ϕ are connected and hence the preimage $W := \phi^{-1}(\bar{X} \setminus X)$ is a connected subvariety. By renumeraling divisors E_i we can assume that a curve Γ_i is an image $\phi(E_i \setminus W)$. Note that $\phi^{-1}(\Gamma_i) \subset Z$ meets the variety W only at one point, it follows that $\Gamma_i = \phi(\mathbb{P}^1 \setminus \{pt\}) \cong \phi(\mathbb{K})$. \square

3. Uniruledness

In this section, we recall the definition of a \mathbb{K} -uniruled variety and we give a characterization of such a variety. The theorem we prove is an affine counterpart of Proposition IV.1.3 from the book [8].

Definition 3.1. We say that an affine irreducible variety X is \mathbb{K} -uniruled if for all $x \in X$ there exists a non-constant morphism $\varphi_x : \mathbb{K} \rightarrow X$ such that $\varphi_x(0) = x$.

In other words it means that it is a sum of parametric curves. If the considered field \mathbb{K} is uncountable we can look at a \mathbb{K} -uniruled variety as dominated by a cylinder. Indeed, we have the following characterization of uniruledness for uncountable fields.

Theorem 3.1. Let X be an affine irreducible variety of dimension n , and assume that \mathbb{K} is uncountable. Then the following conditions are equivalent:

- (1) X is \mathbb{K} -uniruled.
- (2) There exists an open subset $V \subset X$, such that for all $x \in V$ there exists a non-constant morphism $\varphi_x : \mathbb{K} \rightarrow X$ for which $\varphi_x(0) = x$.
- (3) There exist a variety R of dimension $n - 1$ and a dominant morphism $\phi : \mathbb{K} \times R \rightarrow X$.

Proof. Since X is an affine variety we can assume that $X \subset \mathbb{K}^m$ for some m . Implication $1 \Rightarrow 2$ is trivial. To prove implication $2 \Rightarrow 3$ let $S_d := \{\varphi : \mathbb{K} \rightarrow \mathbb{K}^m \text{ such that } \deg \varphi = d\}$, note that it is a quasi-projective variety since each component of a morphism $\varphi = (\varphi_1, \dots, \varphi_m)$ is a polynomial in one variable t of the form $\varphi_i(t) = \sum_{j=0}^d a_j^i t^j$, and hence for every morphism φ corresponds one point $(a_0^1, \dots, a_d^1, \dots, a_0^m, \dots, a_d^m) \in (\mathbb{K}^{d+1})^m \setminus (\mathbb{K}^d \times \{0\})^m$. Let us consider a variety $\mathcal{S}_d := \{\varphi \in S_d \text{ such that } \varphi(\mathbb{K}) \subset X\}$, it follows that \mathcal{S}_d is closed subset of S_d . Take now the morphism

$$F_d : \mathbb{K} \times \mathcal{S}_d \ni (t, \varphi) \rightarrow \varphi(t) \in X.$$

Since $\mathbb{K} \times \mathcal{S}_d$ is an algebraic set then the image $F_d(\mathbb{K} \times \mathcal{S}_d) =: X_d$ is constructible. Let us notice that thanks to the Baire's Theorem for the Zariski topology ([11, Proposition 9.4]) there exists at least one natural number d such that $\overline{X_d} = X$. For this d the morphism $F_d : \mathbb{K} \times \mathcal{S}_d \rightarrow X$ is dominant. Let $Y \subset \mathcal{S}_d$ be an affine irreducible smooth component such that the restriction

$$\tilde{\phi} := F_d|_{\mathbb{K} \times Y} : \mathbb{K} \times Y \rightarrow X$$

is also dominant. Let us embed Y into \mathbb{K}^N for some $N \in \mathbb{N}$ and put $r = \dim Y$.

For a sufficiently general point $x \in \tilde{\phi}(\mathbb{K} \times Y)$ we have $\dim \tilde{\phi}^{-1}(x) = r + 1 - n$. From the construction of F_d the fiber $F := \tilde{\phi}^{-1}(x)$ does not contain any line of type $\mathbb{K} \times \{y\}$. In particular, the image F' of the fiber F under projection $\mathbb{K} \times Y \rightarrow Y$ has also dimension $r + 1 - n$.

Now take a general linear subspace $L \subset \mathbb{K}^N$ of dimension $N + n - r - 1$. We have that the dimension of $L \cap F'$ is 0. Let R be an irreducible component of the intersection $L \cap Y$ (which has non-empty intersection with F'), it has dimension $n - 1$. Let us consider the mapping $\phi : \mathbb{K} \times R \rightarrow X$ induced by the mapping $\tilde{\phi}$. It is easy to see that the mapping ϕ has one fiber of dimension 0 and since $\dim \mathbb{K} \times R = \dim X$ this mapping is dominant. This establishes implication $2 \Rightarrow 3$.

Proof of implication $3 \Rightarrow 1$: We can reduce our proof to the case where X is a surface. Indeed, let $\phi : \mathbb{K} \times R \rightarrow X$ be a dominant morphism. Let $\gamma \subset X$ be an irreducible curve passing through the point $x \in X$, such that $l := \gamma \setminus \{x\} \subset \phi(\mathbb{K} \times R)$. Of course we can always find such a curve because ϕ is dominant. Let us take a curve $C \subset \phi^{-1}(l)$ such that the restriction map $\phi|_C : C \rightarrow l$ is dominant. Define now $\Gamma := p_2(C)$, where $p_2 : \mathbb{K} \times R \rightarrow R$ is a projection. If $\dim \Gamma = 0$ then $C \subset \mathbb{K} \times \{r_1, \dots, r_k\}$ for some points $r_i \in R$. It means that there exists a point r_i such that $l = \phi|_{\mathbb{K} \times \{r_i\}}(\mathbb{K} \times \{r_i\})$ hence γ is the parametric curve we were looking for. On the other hand if Γ is a curve, consider the restriction map

$$\tilde{\phi} := \phi|_{\mathbb{K} \times \Gamma} : \mathbb{K} \times \Gamma \rightarrow \overline{\phi(\mathbb{K} \times \Gamma)}.$$

It is clear that this is a dominant morphism from the cylinder to an affine subset of dimension 2. After this remark let us assume that X is a surface. Let Γ be a curve and $\phi : \mathbb{K} \times \Gamma \rightarrow X$ be a dominant morphism. By shrinking Γ if necessary, we can always assume that for any $a \in \Gamma$ the set $\phi(\mathbb{K} \times \{a\})$ is a curve. If $x \in \phi(\mathbb{K} \times \Gamma)$ then we can take as a parameterization $\varphi_x : \mathbb{K} \rightarrow X$ the restriction map $\varphi_x(t) := \phi|_{\mathbb{K} \times \{y\}}(t + a, y)$, where $(a, y) = \phi^{-1}(x)$. In the other case, if $x \in X \setminus \phi(\mathbb{K} \times \Gamma)$, consider the projective closure of ϕ , that is a map $\bar{\phi} : \mathbb{P}^1 \times \bar{\Gamma} \rightarrow \bar{X}$. Let a_1, \dots, a_k be all points of irregularity of $\bar{\phi}$. Observe that $a_i \in (\{\infty\} \times \bar{\Gamma}) \cup (\mathbb{P}^1 \times (\bar{\Gamma} \setminus \Gamma))$ for all $i \in \{1, \dots, k\}$. There is a finite number of blowing-ups π_1, \dots, π_m such that the following diagram is commutative:

$$\begin{array}{ccc} Y & & \\ \pi \downarrow & \searrow \Phi & \\ \mathbb{P}^1 \times \bar{\Gamma} & \xrightarrow{\bar{\phi}} & \bar{X} \end{array}$$

where Φ is a morphism and $\pi = \pi_1 \circ \dots \circ \pi_m$. Let $T \subset Y$ be a tree of resolution of points of indeterminacy. Notice that we have the following equality

$$X \setminus \phi(\mathbb{K} \times \Gamma) = \Phi(T \setminus W),$$

where $W := \Phi^{-1}(\bar{X} \setminus X)$, thanks to Proposition 2.1, is connected as the complement of a semi-affine set. Take an exceptional curve $E \subset T$ of π such that $x \in \Phi(E)$. Using the same argument as in the proof of Proposition 2.2 we see that $W \cap E = \{y\}$ and consequently $x \in \Phi(\mathbb{P}^1 \setminus \{y\})$. This completes the proof. \square

4. Main result

Theorem 4.1. *If the Jelonek set of a generically finite map $f : \mathbb{K}^n \rightarrow \mathbb{K}^m$ is not empty then it is a \mathbb{K} -uniruled variety of pure dimension $n - 1$.*

Here, we show only that this set is \mathbb{K} -uniruled. The proof that the set J_f is of pure dimension $n - 1$ was given by Jelonek in [5]. At the beginning we give a proof of a special case.

Lemma 4.1. *Let V be an affine irreducible surface such that $\mathbb{K}^2 \subset V$ is an open dense subset and let $f : V \rightarrow \mathbb{K}^m$ be a generically finite map. Then the Jelonek set J_f is a union of parametric curves or the empty set.*

Proof. We factorize the map f by the graph map and the canonical projection. That is $f = \pi \circ g$, where

$$g : V \ni x \rightarrow (x, f(x)) \in \text{graph}(f) \subset V \times \mathbb{K}^m.$$

Since the set J_f is equal to the image under the projection π of the set

$$Y = \overline{\text{graph}(f)} \setminus \text{graph}(f) \subset \mathbb{P}^2(V) \times \mathbb{K}^m,$$

it is enough to show that the set Y is \mathbb{K} -uniruled. This follows from Proposition 2.2 applied to $\text{graph}(f)$. Indeed, by normalization we can assume that it is normal variety, then as $\text{graph}(f)$ is isomorphic to the variety V which contains a copy of \mathbb{K}^2 as a dense subset, we have that $W := \overline{\text{graph}(f)} \setminus \mathbb{K}^2$ is a reunion of parametric lines. Since $Y \subset W$ and in our situation $\dim J_f = 1$ we conclude that Y is \mathbb{K} -uniruled. \square

To prove the main theorem, we will need the following facts. One about extensions of maps and a second about embedding of curves.

Theorem 4.2. *Let X be an affine variety of dimension at most n and let $Z \subset X$ be a closed subset. Then for each quasi-finite map $f : Z \rightarrow \mathbb{K}^n$ there exists a quasi-finite extension $F : X \rightarrow \mathbb{K}^n$.*

Lemma 4.2. *Let γ be an irreducible curve. Then there exist a finite number of points $x_1, \dots, x_k \in \gamma$ and an embedding $\iota : (\gamma \setminus \{x_1, \dots, x_k\}) \rightarrow \mathbb{K}^2$ such that the image of ι is a closed subset of the target space.*

For both proofs see [4].

Proof of Theorem 4.1. Let $y \in J_f$ be a point, we have to find a parametric curve in the set J_f passing through y . First, we take a factorization of the map f as usual by a composition of its graph map g and the canonical projection π restricted to $\text{graph}(f) \subset \mathbb{P}^n \times \mathbb{K}^m$. Let $z = (x, y)$ be a point in the fiber $\pi^{-1}(y)$ such that $z \in \overline{\text{graph}(f)} \setminus \text{graph}(f)$. Let Γ be a

curve contained in the graph of the map f , such that $x \in \overline{\Gamma}$, denote then by γ the image of Γ under the projection to \mathbb{K}^n . After removing, if necessary, some finite number of points x_1, \dots, x_k we can embed $\tilde{\gamma} := \overline{\gamma} \setminus \{x_1, \dots, x_k\}$ into the space \mathbb{K}^2 . Let ι be such an embedding. Now we can extend the map $\iota^{-1} : \iota(\tilde{\gamma}) \rightarrow \gamma$ to a quasi-finite map $\phi : \mathbb{K}^2 \rightarrow \mathbb{K}^n$. Due to Grothendieck's version of the Zariski's Main Theorem there exists an affine variety V such that $\mathbb{K}^2 \subset V$ is a dense subset, and a finite map $\Phi : V \rightarrow \mathbb{K}^n$ such that $\Phi|_{\mathbb{K}^2} = \phi$. Put $\varphi := f \circ \Phi$, for such defined mapping we have $y \in J_\varphi \subset J_f$. Indeed, the map f is not finite on the curve γ and hence for every neighborhood U of the point y the map f is not finite on the set $\Phi(\varphi^{-1}(U)) \subset f^{-1}(U)$. Now, since for every subset $U \subset \mathbb{K}^m$ we have $\varphi|_{\varphi^{-1}(U)} = f|_{f^{-1}(U)} \circ \Phi|_{\varphi^{-1}(U)}$ then if U is an open neighborhood of the point y the restriction map $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \rightarrow U$ cannot be finite. On the other hand, for a point $u \in \mathbb{K}^m \setminus J_f$ there exists an open affine neighborhood U such that $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is a finite map, hence also $\varphi|_{\varphi^{-1}(U)}$ is finite as a composition of finite maps, which proves that $J_\varphi \subset J_f$. Finally, thanks to Lemma 4.1 there is a parametric curve in J_φ and hence in J_f passing through the point y . This ends the proof. \square

5. Final remarks

We say that an affine irreducible variety X of dimension n is separably \mathbb{K} -uniruled if there exist a variety R of dimension $n - 1$ and a separable dominant morphism $\phi : \mathbb{K} \times R \rightarrow X$.

If a variety is separably \mathbb{K} -uniruled it cannot be of general type. As a consequence the Jelonek set in characteristic 0 is never of general type because all maps are separable. In positive characteristic arises a natural question: *Is the Jelonek set for a separable map necessarily separably \mathbb{K} -uniruled?* The answer is negative. We construct an example using some methods of Jelonek.

At the beginning let us recall a criterium for a map to be separable (compare [9, VIII,5]). For the convenience of the reader we give here an elementary proof of a fact which in general version is proved in Matsumura's book ([10, 27.B, Theorem 59]).

Proposition 5.1. *A dominant polynomial map $f = (f_1, \dots, f_n) : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is separable if and only if $Jac(f) \neq 0$.*

Proof. Assume that $Jac(f) \neq 0$. Let x_1, \dots, x_n be coordinates in \mathbb{K}^n . We have to show that the field extension $\mathbb{K}(f) \subset \mathbb{K}(x_1, \dots, x_n)$ is separable. For this it is enough to show that any derivation D on $\mathbb{K}(x_1, \dots, x_n)$ which vanishes on $\mathbb{K}(f)$ is equal to zero everywhere. Let $F_i := f_i - f_i(x_1, \dots, x_n) \in \mathbb{K}(f)[X]$. Since

$$0 = DF_i = \sum_j \frac{\partial f_i}{\partial x_j} Dx_j,$$

we obtain a linear system of equations with non zero determinant, it means that $Dx_j = 0$. If $Jac(f) = 0$ we can easily construct a derivation which vanishes on $\mathbb{K}(f)$ but not on $\mathbb{K}(x_1, \dots, x_n)$. \square

The next proposition is a generalization of the previous fact to the case of affine varieties.

Proposition 5.2. *Let X, Y be affine irreducible varieties. A generically finite dominant map $f : X \rightarrow Y$ is separable if and only if there exists a smooth point $x \in X$ such that its derivative function*

$$d_x f : T_x X \rightarrow T_{f(x)} Y,$$

is an isomorphism.

Proof. Let x_1, \dots, x_n be coordinates on X . We can assume that $Y \subset \mathbb{K}^m$ and hence $f = (f_1, \dots, f_m)$. If $f : X \rightarrow Y$ is a separable map then, since every separable extension has a primitive element ([9, V.4; Theorem 4.6]), there exists an element $c \in \mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]$ such that

$$\mathbb{K}(f, c) = \mathbb{K}(X). \quad (5.1)$$

Let $Z = \text{Spec}(\mathbb{K}[f, c])$. We can treat Z as a subvariety of $Y \times \mathbb{K}$ which is birational to X by the induced map from equality (5.1), it means

$$\varphi : X \ni x \mapsto (f(x), c(x)) \in Z.$$

Let

$$Q(f, t) = t^k + a_1(f)t^{k-1} + \dots + a_k(f)$$

be an irreducible polynomial defining Z , where $a_i \in \mathbb{K}[f]$ and $Q(f, c) = 0$. The following diagram is commutative:

$$\begin{array}{ccc} T_x X & \xrightarrow{(df, dc)} & T_{\varphi(x)} Z \\ & \searrow df & \downarrow \\ & & T_{f(x)} Y. \end{array}$$

Since the extension $\mathbb{K}(f) \subset \mathbb{K}(X)$ is separable we have inequality $\partial Q / \partial t \neq 0$. It means that for a generic z we have

$$d_z c = - \frac{\sum_i (\partial Q / \partial f_i) d_z f_i}{(\partial Q / \partial t)}$$

and in consequence dc belongs to the space generated by $df_i, i = 1, \dots, m$ on some non-empty open set. It means that df is an isomorphism on some dense set.

Assume now that there is a smooth point $x \in X$ such that $d_x f$ is an isomorphism. We will show that every derivation of $\mathbb{K}(X)$ which vanishes on $\mathbb{K}(f)$ is trivial, which proves that the field extension $\mathbb{K}(f) \subset \mathbb{K}(X)$ is separable. Due to the assumptions the following map between cotangent spaces

$$df^* : T_{f(x)}^* Y \ni d\alpha \mapsto d(\alpha \circ f) \in T_x^* X,$$

is an isomorphism. So, we have that

$$\langle dx_1, \dots, dx_n \rangle = \langle df_1, \dots, df_m \rangle, \quad (5.2)$$

where by $\langle a, \dots, b \rangle$ we mean the space generated by a, \dots, b . Let $h \in \mathbb{K}(X)$ then

$$\text{grad } h = \sum_{i=1}^n a_i \text{grad } f_i, \quad (5.3)$$

where $\text{grad } f = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$. Suppose now that D is a trivial derivation on $\mathbb{K}(f)$. The Equalities (5.2) and (5.3) show that

$$Dh = (\text{grad } h | Dx) = \sum a_i (\text{grad } f_i | Dx) = 0,$$

where $(|)$ is the scalar product, and Dx a vector (Dx_1, \dots, Dx_n) . This ends the proof. \square

At the end let us notice that we are able to extend any polynomial map to a separable one.

Proposition 5.3. *Let $X \subsetneq V \subset \mathbb{K}^N$ be affine varieties, where V is irreducible of dimension n . Then for each polynomial map $f : X \rightarrow \mathbb{K}^m$, where $n \leq m$, there exists a separable extension $F : V \rightarrow \mathbb{K}^m$ to the whole ambient variety. Moreover if a map f is finite then this extension map F can be chosen to be also finite.*

Proof. Let (g_1, \dots, g_k) be an ideal of the variety X and x_1, \dots, x_n the coordinates of V . Consider the map

$$\phi = (\tilde{f}, g_1, \dots, g_k, g_i \pi_j)_{(i,j)} : V \rightarrow \mathbb{K}^{m+k+kn},$$

where \tilde{f} is a polynomial extension of the map f to the variety V , and $\pi_j(x_1, \dots, x_n) = x_j$. Since we can calculate x_i using g_j the map ϕ is birational onto its image. Hence it is a separable map. Let $y \in \phi(V)$ be a smooth point. Let \mathcal{V} be a subspace of \mathbb{K}^{m+k+kn} of dimension $k + kn$ such that:

- (1) the intersection $\mathcal{V} \cap T_y \phi(V)$ is transverse,
- (2) the intersection $\overline{\mathcal{V}} \cap \overline{\phi(V)} \cap \Pi_\infty$ is empty, where Π_∞ is the hyperplane at infinity.

Set $\pi : \mathbb{K}^{m+n+kn} \rightarrow \mathbb{K}^m$ to be the projection along the direction of the subspace \mathcal{V} . Thanks to the previous proposition the restriction $p_2 := \pi|_{\phi(V)}$ is separable. Hence $F = p_2 \circ \phi$ is the separable map we were looking for. Indeed, $F|_X = f$ is a composition of separable maps, so it is also separable.

Let us remark at the end that if f is finite map we can slightly modify the construction of ϕ . In this case the extension $\mathbb{K}[f] \subset \mathbb{K}[X]$ is integral. Let

$$h_i(x_i) = x_i^{s_i} + x_i^{s_i-1} u_{i1} + \dots + u_{is_i}$$

be a polynomial of integral dependence of x_i on $\mathbb{K}[f]$ for $i = 1, \dots, n$. Consider now the map

$$\phi = (\tilde{f}, g_1, \dots, g_k, g_i \pi_j, h_1 \circ \pi_1, \dots, h_n \circ \pi_n)_{(i,j)},$$

which by construction is finite. We can take a subspace \mathcal{V} like in the first part of the proof to get a separable and finite projection $p_2 : \phi(V) \rightarrow \mathbb{K}^m$. Hence $F := p_2 \circ \phi$ is finite as the composition of finite maps. \square

Now we are ready to give an example of a separable map for which the Jelonek set fails to be separably \mathbb{K} -uniruled.

Proposition 5.4. *Let \mathbb{K} be a field of characteristic p . Then there is a separable generically finite map $f : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ such that the set J_f is not a separably \mathbb{K} -uniruled surface.*

Proof. In fact, let $Z \subset \mathbb{K}^3$ be a hypersurface given by the equation $z^p - f(x, y) = 0$, where f is a sufficiently general polynomial of two variables. Since there is a parametrization $\phi : \mathbb{K}^2 \ni (u, v) \mapsto (u^p, v^p, \sqrt[p]{f(u^p, v^p)}) \in Z$ it is the rational \mathbb{K} -uniruled hypersurface but, according to the construction of Kollár [7], it is not separably \mathbb{K} -uniruled. Now we will construct an example for characteristic $p \neq 2$ of a generically-finite map $f : \mathbb{K}^3 \rightarrow \mathbb{K}^3$ such that the Jelonek set J_f is equal to Z (for $p=2$ one has to modify a little the construction). Let $W = \{(u, v, w) \in \mathbb{K}^3 \text{ such that } w^2 - uv - 1 = 0\}$ and let $V = \mathbb{K} \times W$.

We embed an affine space \mathbb{K}^3 into a variety V by the following map

$$\iota : \mathbb{K}^3 \ni (x, y, z) \rightarrow (z, x(2 + xy), y, 1 + xy) \in V.$$

By the construction the set $X = V \setminus \iota(\mathbb{K}^3)$ is isomorphic to the space \mathbb{K}^2 . We have a dominant map $\varphi : X \rightarrow Z$. Now thanks to Proposition 5.3 we can extend a map φ to a separable map $\Phi : V \rightarrow \mathbb{K}^3$ and finally we define a map f which we were looking for as the composition $\Phi \circ \iota$. It is easy to check that the map f is separable and the Jelonek set J_f is equal to the hypersurface Z . \square

This proposition shows that we cannot expect much more if we consider separable maps.

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